

A Ramanujan congruence analogue for Han's hook-length formula mod 5, and other symmetries

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Abstract

This article considers the eta power $\prod (1 - q^k)^{b-1}$. It is proved that the coefficients of $\frac{q^n}{n!}$ in this expression, as polynomials in b , exhibit equidistribution of the coefficients in the nonzero residue classes mod 5 when $n = 5j + 4$. Other symmetries, as well as symmetries for other primes and prime powers, are proved.

1 Introduction

The Han/Nekrasov-Okounkov hook length formula, below, was discovered independently and by significantly different means, first by Nekrasov and Okounkov ([7]) and shortly thereafter by Guo-Niu Han ([5]).

$$\prod_{k=1}^{\infty} (1 - q^k)^{b-1} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h_{ij} \in \lambda} \left(1 - \frac{b}{h_{ij}^2}\right) \quad (1)$$

The formula relates powers of the partition function $\prod \frac{1}{1-q^k}$ or, inversely, the eta function $q^{\frac{1}{24}} \prod (1 - q^k)$, to a sum of products over the hooklengths of all partitions. For any given n , the coefficient on q^n is clearly a polynomial in the variable b .

Congruences for the numerical values of particular powers of the partition function have been studied (see [1],[2],[3], and [4], among others; the first is a survey of related literature). The object of this paper is to study the values of the coefficients of q^n as polynomials in the b . Normalized by $\frac{1}{n!}$ to make these coefficients integer polynomials, we find many pleasing results on the distribution and arrangement of the coefficients of these polynomials.

The most classical congruences for powers of the partition function are Ramanujan's congruences for the partition function itself, i.e. the case $b = 0$. Of particular interest to us is the first of these, that

$$p(5k + 4) \equiv 0 \pmod{5}$$

for any integer k . We show an analogue of this result for the hook length formula, namely,

Theorem 1. For $n = 5k + 4$, the integer polynomial $p_n(b)$ defined by $\prod_{k=1}^{\infty} (1 - q^k)^{b-1} = \sum_{n=0}^{\infty} \frac{q^n}{n!} p_n(b)$ has coefficients for which the following symmetries hold:

- The nonzero residues equally populate the residue classes 1, 2, 3, and 4 mod 5.
- These residues appear in groups of four as a rotation of the list (2, 4, 3, 1).
- The coefficients of the k terms of lowest degree are all 0 mod 5. (There may be others.)

Remarks. Concerning integrality, while it may not be immediately obvious that the polynomials are integral, this is easily shown by deriving a recurrence for the polynomials with the $q \frac{\partial}{\partial q} \log$ technique of Herb Wilf's generatingfunctionology ([8]), or referring to Corollary 2.3 of ([6]).¹

Equidistribution is only the first of many beautiful symmetries exhibited by these coefficients. Noting from the second clause of the theorem that the $4j$ -th coefficient identifies coefficients through $4j + 3$, we can display every 4th coefficient starting from the $k + 1$ position (the first k being 0). If we do this, leaving a space for zeroes to make the structure more visually apparent, we obtain the striking triangle on the next page. This triangle is exactly Pascal's triangle, multiplied by 2 and an alternating sign, reduced mod 5.

The structure of this paper is as follows. In the next section we prove Theorem 1. In Section 3 we show additional symmetries, which are corollaries of the proof of Theorem 1 and well-known facts on binomial coefficients. We then consider other prime and prime power arithmetic progressions. In Section 4 we discuss a few open questions on the combinatorics of these polynomials, which will hopefully motivate future work in this study.

¹Because this recurrence is useful for calculating the polynomials discussed in this paper, we state it here for the benefit of readers. With $p_0(b) = 1$, and σ_1 the divisor function, for $n \geq 1$ we have

$$p_n(b) = (n-1)!(b-1) \sum_{m=1}^n \frac{-\sigma_1(m)p_{n-m}(b)}{(n-m)!}.$$

A little more work can derive an expression for the coefficients individually, but the terms for the coefficient of b^k in p_n are indexed by walks in an $n \times k$ lattice and so grow exponentially in number.

3

2 Proof of the main theorem

We begin by expanding the product out using the generalized binomial theorem. Recall that for any value x , including the indeterminate in $\mathbb{C}[x]$, we can define the generalized binomial coefficient with a whole number k as $\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$.

We use the notation $\mathbf{e} \vdash n$ to mean that \mathbf{e} is a partition of n , and write partitions in the frequency notation $\mathbf{e} = 1^{e_1}2^{e_2}3^{e_3}\dots$ to denote the partition in which 1 occurs e_1 times, 2 occurs e_2 times, etc. It is understood that the e_i are nonnegative integers and that only finitely many of the e_i are nonzero.

Expanding with the generalized binomial theorem, we first obtain

$$\begin{aligned}
\prod_{j=0}^{\infty} (1 - q^j)^{b-1} &= \prod_{j=0}^{\infty} \frac{1}{(1 - q^j)^{1-b}} \\
&= \prod_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{1-b+k-1}{k} (q^j)^k = \prod_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{k-b}{k} (q^j)^k \\
&= \prod_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(k-b)(k-b-1)\cdots(k-b-k+1)}{k!} (q^j)^k \\
&= \prod_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1-b)(2-b)\cdots(k-b)}{k!} (q^j)^k \\
&= \sum_{n=0}^{\infty} q^n \sum_{\mathbf{e} \vdash n} \prod_{j=1}^{\infty} \frac{(1-b)(2-b)\cdots(e_j-b)}{e_j!} \\
&= \sum_{n=0}^{\infty} \frac{q^n}{n!} \sum_{\mathbf{e} \vdash n} \frac{n!}{e_1!e_2!\dots} \prod_{j=1}^{\infty} (1-b)(2-b)\cdots(e_j-b).
\end{aligned}$$

Note that $\frac{n!}{e_1!e_2!\dots}$ is not a multinomial coefficient since $e_1 + e_2 + \dots \neq n$ except in the trivial case $e_1 = n$.

Considering the various powers of b in the final polynomials, one polynomial for each e_i , we see that this expression is equal to

$$\sum_{n=0}^{\infty} \frac{q^n}{n!} \sum_{\mathbf{e} \vdash n} \frac{n!}{e_1!e_2!\dots} \prod_{j=1}^{\infty} \sum_{t=0}^{e_j} (-b)^t \sum_{\substack{S=\{s_1,\dots,s_{e_j-t}\} \\ S \subseteq \{1,\dots,e_j\}}} s_1 s_2 \cdots s_{e_j-t}.$$

The products on the far right are every possible selection of $e_j - t$ distinct elements from the set $\{1, 2, \dots, e_j\} =: [e_j]$. When we bring the factor of $\frac{1}{e_j!}$ back into this product, we exchange this for a sum over their complements, all possible denominators consisting of products of t distinct elements from $\{1, 2, \dots, e_j\}$.

Multiplying these together, we get

$$\prod_{k=1}^{\infty} (1 - q^k)^{b-1} = \sum_{n=0}^{\infty} \frac{q^n}{n!} \sum_{\substack{\mathbf{e} \vdash n \\ \mathbf{e} = 1^{e_1} 2^{e_2} \dots}} n! \left[\sum_{t=0}^n (-b)^t \left(\sum \frac{1}{s_1 \dots s_t} \right) \right]$$

where the last sum runs over unordered t -tuples of distinct elements chosen from the multiset $\{1, 2, \dots, e_1, 1, 2, \dots, e_2, \dots\}$.

Thus, the coefficient of $(-b)^t$ in $p_n(b)$ is

$$n! \sum_{\substack{\mathbf{e} \vdash n \\ \mathbf{e} = 1^{e_1} 2^{e_2} \dots}} \left(\sum \frac{1}{s_1 \dots s_t} \right) .$$

Our task is now to determine the residue class of this integer mod 5 when $n = 5k + 4$.

In any given term, if the power of 5 that divides $n!$ is not fully canceled by elements of the product $s_1 \dots s_t$, that term will contribute 0 to the residue class of the sum mod 5. It is possible for this to occur if $e_1 \geq 5k$, and among the s_i are $5, 10, 15, \dots, 5k$ chosen from the first part of the multiset, $\{1, 2, \dots, e_1, (\dots)\}$. If $e_1 < 5k$, since all other parts are of size at least 2, it is clear that the deficit in available entries can never be fully made up by elements chosen from $\{1, \dots, e_j\}$ with $j > 1$. For example, if $e_2 \geq 5$, then $e_1 < 5k - 5$, etc.

Hence the only partitions \mathbf{e} that can possibly contribute to the sum mod 5 are:

$$1^{5k+4} \quad , \quad 1^{5k+2} 2^1 \quad , \quad 1^{5k+1} 3^1 \quad , \quad 1^{5k} 4^1 \quad , \quad 1^{5k+4} 2^2 .$$

Among the contributions from these partitions, only those terms in which k of the elements are assigned to $5, 10, \dots, 5k$ can possibly contribute to the residue of the sum.

Thus, if $t < k$, no terms contribute a nonzero value. If $t = k$, then exactly the one term $\frac{(5k+4)!}{5 \cdot 10 \cdot \dots \cdot 5k}$ contributes. This happens once in each of the five possible partitions, so the total is 0 mod 5. This proves the last clause of Theorem 1.

Suppose now that $t = k + m$, $m > 0$, and that k of the elements of $\{s_1, \dots, s_t\}$ are $5, 10, \dots, 5k$. There remain the contributions constructed from choosing m values from the remaining places in the five possible partitions. Since we are only concerned with the residue of the result mod 5, we need only classify the multiset of available choices in each partition by their residues mod 5:

- $1^{5k+0} 4^1$: the set $\{1, 2, 3, 4\}$ repeated k times, plus the set $\{1\}$
- $1^{5k+0} 2^2$: the set $\{1, 2, 3, 4\}$ repeated k times, plus the set $\{1, 2\}$
- $1^{5k+1} 3^1$: the set $\{1, 2, 3, 4\}$ repeated k times, plus sets $\{1\}$ and $\{1\}$
- $1^{5k+2} 2^1$: the set $\{1, 2, 3, 4\}$ repeated k times, plus $\{1, 2\}$ and $\{1\}$
- 1^{5k+4} : the set $\{1, 2, 3, 4\}$ repeated k times, plus another set $\{1, 2, 3, 4\}$

If the m additional elements are all chosen from the first set $\{1, 2, 3, 4\}$ repeated k times, the same choices may be made in any of the five partitions. We group these to form a 0 contribution to the residue of the sum.

We again group all those terms where $m - 1$ of the choices are made in the first k sets of 4, and the first 1 of the additional sets is chosen.

We again group all those terms where $m - 1$ of the choices are made in the first k sets of 4, and the 2 is chosen – multiplying the value obtained from the first $m - 1$ choices by $2^{-1} \bmod 5$, or 3 – or the second 1 of the partition $1^{5k+1}3^1$ is chosen. There are then three instances where the contribution is multiplied by 3, and one multiplied by 1, for a total multiplier of 10. These thus contribute 0 residue mod 5.

We again group all those terms where $m - 1$ of the choices are made in the first k sets of 4, and the second 1 of $1^{5k+2}2^1$ is chosen, or the 4 of 1^{5k+4} .

Among those terms in which $m - 1$ of the choices are made in the first k sets of 4, we are left with those in which the 3 of the 1^{5k+4} is chosen, multiplying the contribution of these elements by 2 mod 5.

Now suppose $m - 2$ of the choices are made in the first k sets of 4. We can form the following large group that contributes 0 mod 5: any choice of 1 and 2 from the same set (there are 3 of these), or the 2 and the second 1 of $1^{5k+2}2^1$, the pairs of 1s from $1^{5k+1}3^1$ and $1^{5k+2}2^1$, and of 1^{5k+4} the 1 and 3, the 1 and 4, the 2 and 4, and the 3 and 4. Totaling the inverses of these, we obtain 0 mod 5.

Among those terms in which $m - 2$ of the choices are made in the first k sets of 4, we are left with those in which the 2 and 3 of the 1^{5k+4} is chosen, multiplying the contribution of these elements by 1 mod 5.

If $m - 3$ choices are made in the first k sets of 4, then we have 3 choices to make from the remaining elements. The 4 possible choices from $1^{5k+4} - \{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$ – group to add 0 mod 5. We are left with the choice of 1, 2, and 1 from $1^{5k+2}2^1$, multiplying the contribution of these choices by 3.

If $m - 4$ choices are made in the first k sets of 4, the only remaining possibility is all of 1, 2, 3, and 4 from 1^{5k+4} , multiplying these contributions by 4.

For any given $t = k + m$, the sum is thus a linear combination of the residues mod 5 of the various ways to choose $m - 1$ through $m - 4$ elements of the multiset $\{1, 2, 3, 4\}^k$ as denominators. We have therefore reduced the question of determining the residue class of the coefficient of b^t in $p_n(b)$ to calculating the residue class mod 5 of the following sum:

$$\begin{aligned}
& \frac{(-1)^t (5k+4)!}{5 \cdot 10 \cdots 5k} \left(\sum_{r_1+r_2+r_3+r_4=m-1} \frac{1}{3} \binom{k}{r_1} \binom{k}{r_2} \binom{k}{r_3} \binom{k}{r_4} \frac{1}{1} \frac{1}{2} \frac{1}{3} \frac{1}{4} \right. \\
& + \sum_{r_1+r_2+r_3+r_4=m-2} \frac{1}{6} \binom{k}{r_1} \binom{k}{r_2} \binom{k}{r_3} \binom{k}{r_4} \frac{1}{1} \frac{1}{2} \frac{1}{3} \frac{1}{4} \\
& + \sum_{r_1+r_2+r_3+r_4=m-3} \frac{1}{2} \binom{k}{r_1} \binom{k}{r_2} \binom{k}{r_3} \binom{k}{r_4} \frac{1}{1} \frac{1}{2} \frac{1}{3} \frac{1}{4} \\
& \left. + \sum_{r_1+r_2+r_3+r_4=m-4} \frac{1}{24} \binom{k}{r_1} \binom{k}{r_2} \binom{k}{r_3} \binom{k}{r_4} \frac{1}{1} \frac{1}{2} \frac{1}{3} \frac{1}{4} \right) .
\end{aligned}$$

The binomial coefficients arise from the fact that, if m choices are made of which r_1 are 1s, then there are $\binom{k}{r_1}$ ways to make these choices.

Because all of these coefficients are integers, we can treat this sum as arithmetic in the residue classes mod 5: the coefficient of b^t in $p_n(b)$, where $t = k + m$ and $n = 5k + 4$, thus becomes

$$(-1)^{m+1} \left(\sum_{\substack{r_1+r_2+r_3+r_4=m-c \\ c=1,2,3,4}} a_c \binom{k}{r_1} \binom{k}{r_2} \binom{k}{r_3} \binom{k}{r_4} 1^{r_1} 3^{r_2} 2^{r_3} 4^{r_4} \right)$$

where $a_1 = 2$, $a_2 = 1$, $a_3 = 3$, and $a_4 = 4$.

We now wish to evaluate this sum, whose symmetries over the range of m range yield the behaviors of Theorem 1 and the additional symmetries visible in the triangle. It will be useful to evaluate the sum for a general prime p , as follows:

Lemma 1. *For p a prime,*

$$\begin{aligned}
& \sum_{r_1+\cdots+r_{p-1}=(p-1)s+c} \binom{k}{r_1} \cdots \binom{k}{r_{p-1}} 1^{r_1} 2^{r_2} \cdots (p-1)^{r_{p-1}} \\
& \equiv \begin{cases} (-1)^s \binom{k}{s} \mod p & c = 0 \\ 0 \mod p & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. (Note that we assign bases to exponents of their own index, for convenience. The sum is symmetric under this exchange.)

The left-hand side is the coefficient of $q^{(p-1)s+c}$ in the product

$$\begin{aligned}
(1+q)^k \cdots (1+(p-1)q)^k &= (1+(\sum_{i=1}^{p-1} i)q + (\sum_{\substack{i,j=1 \\ i < j}}^{p-1} ij)q^2 + \cdots + (p-1)!q^{p-1})^k \\
&\equiv (1 - q^{p-1})^k \mod p .
\end{aligned}$$

The middle terms in the identity vanish (observe the effect of permuting the entries by a nontrivial multiplication in \mathbb{Z}_p) and $(p-1)! \equiv -1 \mod p$.

p , giving the last line. The coefficient of $q^{(p-1)s}$ in the latter expression is exactly $(-1)^s \binom{k}{s}$, and all other coefficients are 0. \square

This proves the claims of the first two clauses of the theorem. The coefficients of b^t are sums of multiples of four consecutive terms of the type evaluated in Lemma 1, only one of which may be nonzero mod 5. The corresponding coefficients of degree $k + 4s + 1$, $k + 4s + 2$, $k + 4s + 3$ and $k + 4s + 4$ are this value times 2, 1, 3, and 4, respectively, and an alternating power of -1 with starting parity determined by s .

Thus the rotation is either $(2, -1, 3, -4) = (2, 4, 3, 1)$, or $(-2, 1, -3, 4) = (3, 1, 2, 4)$ times the underlying value. Both are rotations of the list $(2, 4, 3, 1)$. If an underlying nonzero value is $j \bmod 5$, then the list $(2j, 4j, 3j, 1j)$ is a rotation of $(2, 4, 3, 1)$, as claimed. Finally, when the underlying sum is zero mod 5, all 4 terms are 0 mod 5.

This means that those coefficients on b^t which are nonzero rotate through the nonzero residue classes mod 5 in groups of 4, leading to equidistribution of the pattern claimed in Theorem 1. \square

3 Additional Symmetries

We have now evaluated the coefficient of b^{k+1+4m} in $p_{5k+4}(b)$ to be congruent to $2 \cdot (-1)^m \binom{k}{m} \bmod 5$. This tells us that the entries of the triangle possess all the symmetries of Pascal's triangle, when multiplied by 2 mod 5 and an alternating sign:

Corollary 1. *The coefficient of b^{k+1+4m} in $p_{5k+4}(b)$, $0 \leq m \leq k$, is nonzero if and only if $m_i \leq k_i$ for each digit in the 5-ary expansions $(m)_5 = m_0 m_1 \dots$ and $(k)_5 = k_0 k_1 \dots$.*

Corollary 2. *The triangle is self-similar: the apex digits of the triangles at any level are given by the entries of the triangle one level down, and other entries are those of the fundamental triangle times the apex digit.*

Similar corollaries of any of the symmetries of Pascal's triangle reduced mod 5 can be used.

3.1 Other Primes

It is not the case that equidistribution holds for the nonzero residues classes modulo other primes, whether in the -1 arithmetic progression or in those for which Ramanujan-like congruences hold for the partition function. For example, mod 7, $p_6(b) = 7920 - 18144b + 14674b^2 - 5205b^3 + 805b^4 - 51b^5 + b^6$, reducing to $(3, 0, 2, 3, 0, 5, 1)$ and $p_5(b) \equiv (0, 6, 4, 2, 0, 6)$.

This is the case because the coefficients a_c are not as neatly distributed for other primes as they are for modulus 5. However, equidistribution mod p arises for a different reason in the arithmetic progression $-p-1 \bmod p^2$. This is because the binomial coefficients described by Lemma 1 themselves rotate through the nonzero residues mod p .

Theorem 2. *For p prime, $j \geq 0$, if the number of partitions of $p-1$ is not congruent to 1 mod p , the coefficients of $p_{(pj+p-2)p+p-1}(b)$ equinumerously populate the nonzero residue classes mod p for all j , and if it is, the populations are still equinumerous for $j \equiv -2 \bmod p$.*

In order to prove the theorem for all primes, we need to go through the proof of Theorem 1 in greater generality.

The expansion makes no reference to the value of p until we begin determining the residue class of the coefficient

$$n! \sum_{\substack{\mathbf{e} \vdash n \\ \mathbf{e} = 1^{e_1} 2^{e_2} \dots}} \left(\sum \frac{1}{s_1 \dots s_t} \right) .$$

The same logic holds to show that the number of 1s e_1 must be sufficient to cancel all instances of the prime p in $n!$, and so we have a constant set of contributing partitions that can be grouped mod p to give a series of coefficients a_c , $0 \leq c \leq p-1$. The coefficient of b^t in $p_n(b)$, where $n = p(pj + p - 2) + (p - 1)$ and $t = pj + p - 2 + m$, $m \geq 0$, is congruent to

$$(-1)^{m+1} \left(\sum_{\substack{r_1 + \dots + r_{p-1} = m-c \\ c=0,1,\dots,p-1}} a_c \binom{k}{r_1} \dots \binom{k}{r_{p-1}} 1^{r_1} 2^{r_2} \dots (p-1)^{r_{p-1}} \right) .$$

This is a linear combination of terms of the form addressed by Lemma 1, and so mod p the sequence of coefficients of $p_n(b)$, with the initial segment of 0s of length $pj + p - 2$, is given by

$$\begin{aligned} & (0, \dots, 0, -a_0 \binom{pj+p-2}{0}, a_1 \binom{pj+p-2}{0}, \dots, a_{p-2} \binom{pj+p-2}{0}, \\ & -a_{p-1} \binom{pj+p-2}{0} + a_0 \binom{pj+p-2}{1}, -a_1 \binom{pj+p-2}{1}, \dots, -a_{p-2} \binom{pj+p-2}{1}, \\ & a_{p-1} \binom{pj+p-2}{1} - a_0 \binom{pj+p-2}{2}, a_1 \binom{pj+p-2}{2}, \dots, (-1)^{pj+2p-3} a_{p-2} \binom{pj+p-2}{pj+p-2}, \\ & (-1)^{pj+2p-2} a_{p-1} \binom{pj+p-2}{pj+p-2} + (-1)^{pj+p} a_0 \binom{pj+p-2}{pj+p-1} \Big) . \end{aligned}$$

We now examine the sequences $\{(-1)^{c+1} a_c (-1)^s \binom{pj+p-2}{s}\}$, $0 \leq s \leq pj + p - 2$, for $1 \leq c \leq p-2$. We will separately examine the overlapping sums at the edges.

It is a straightforward calculation to show that

Lemma 2. For p a prime, $0 \leq s \leq pj + p - 2$, $s = gp + h$, $0 \leq h < p$,

$$(-1)^s \binom{pj+p-2}{s} \equiv (h+1) \left((-1)^g \binom{j}{g} \right) \pmod{p} .$$

This shows that the sequences $\{(-1)^{c+1} a_c (-1)^s \binom{pj+p-2}{s}\}$ consist of segments of p elements that are either all 0, or are permutations of $\{1, 2, \dots, p-1\}$ followed by a 0. (The last 0 is in the "place" $s = pj + p - 1$.)

With the lemma, the sequence $\{-a_0 (-1)^s \binom{pj+p-2}{s} - a_{p-1} (-1)^{s-1} \binom{pj+p-2}{s-1}\}$, $0 \leq s \leq pj + p - 1$, $s = gp + h$ with $0 \leq h < p$, reduces to

$$\begin{aligned}
& \{-(a_0(h+1) \binom{j}{g} (-1)^g) + a_{p-1}(h) \binom{j}{\lfloor \frac{s-1}{p} \rfloor} (-1)^{\lfloor \frac{s-1}{p} \rfloor}\} \equiv \\
& \{(-1)^0 \binom{j}{0} (-a_0), (-1)^0 \binom{j}{0} (-2a_0 - 1a_{p-1}), (-1)^0 \binom{j}{0} (-3a_0 - 2a_{p-1}), \\
& \dots, (-1)^0 \binom{j}{0} (-pa_0 - (p-1)a_{p-1}), a_0 \binom{j}{1}, (-1)^1 \binom{j}{1} (-2a_0 - a_{p-1}), \\
& \dots, (-1)^j \binom{j}{j} (-pa_0 - (p-1)a_{p-1})\} .
\end{aligned}$$

This is the set of values $(-1)^x \binom{j}{x} (-a_0 + y(-a_0 - a_{p-1}))$ where $0 \leq x \leq j$ and $0 \leq y \leq p-1$. If $a_0 + a_{p-1} \not\equiv 0 \pmod{p}$, this will equally populate all nonzero residue classes mod p each time y ranges over its values. But a_{p-1} is always $-1 \pmod{p}$, and a_0 is the number of partitions of $p-1$, so this happens whenever the number of partitions of $p-1$ is not congruent to $1 \pmod{p}$.

Even when $a_0 + a_{p-1} \equiv 0 \pmod{p}$, by Lemma 2 the sequence will still equally populate the nonzero residue classes mod p as x ranges over its values when $j \equiv -2 \pmod{p}$, giving us the arithmetic progression $-p^2 - p - 1 \pmod{p^3}$. \square

Remarks: The first time the exceptional case happens is at $p = 71$. The size of the populations in the arithmetic progression $-1 - p - p^2 - p^3 - p^4 - \dots \pmod{p^q}$ for any q can be seen, from the argument above, to be divisible by $(p-1)^{q-2}$, or $(p-1)^{q-3}$ in the exceptional case. It is perhaps interesting in this enumerative combinatoric setting that this arithmetic progression is most concisely defined as $\frac{-1}{1-p} \pmod{p^q}$.

4 Open Questions

This sequence of polynomials is related to a core object in enumerative combinatorics, so questions regarding its symmetries should be of wide interest, and many still await investigation.

One of the most interesting questions is whether we can recover the results of the earlier work mentioned on congruences of powers of the partition function, and perhaps extend them, by means as elementary as possible. If we assume knowledge of the fact the the number of partitions of $5k+4$ is itself divisible by 5, it seems likely that the theorems and lemmas in this paper, combined with elementary properties of binomial coefficients, could yield useful theorems, such as the fact that $\prod (1-q^j)^{b-1}$ is divisible by 0 in the arithmetic progression $n = 5k+4$ as long as $b \not\equiv 3 \pmod{5}$. Exploration of these ideas is intended as the immediate followup to this paper.

Are there any other prime progressions $pk + (p-1)$ where equidistribution occurs? If so, how can we find them efficiently? If not, how could this be proved?

What can we say about progressions with composite moduli other than prime powers?

Is it possible to describe an evocative combinatorial object that the coefficients themselves count? Could this description be useful in proofs regarding their properties, or properties of related objects such as multipartitions?

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